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Euclidean Path Integrals

13.1 Euclidean Path Integrals

Besides Feynman's path integral formulation of quantum mechanics (and extended formulations of quantum electrodynamics and other areas, as mentioned earlier), his path integral formulation of statistical mechanics has also proved to be a very useful development. The latter theory however involves *Euclidean path integrals* or *Wiener* type integrals, which rest on a more steady mathematical foundation. Compare remarks in Section 14.2 of the next chapter, for example, and also in Section 16.1 of Chapter 16.

The Euclidean path integral is obtained formally from the Feynman path integral by analytically continuing real time t to imaginary time $-it$. Under this *Wick rotation* $t \to -it$ we declare that $\frac{d}{dt} \to -i\frac{d}{dt}$: $(\frac{dx}{dt})^2 \to (-i\frac{dx}{dt})^2 = -(\frac{dx}{dt})^2$. Then the action functional

$$S(x(t)) \to \int_{t_a}^{t_b} \left[-\frac{m}{2}x'(t)^2 - V(x(t))\right](-i)\,dt = iS_E(x(t)), \tag{13.1.1}$$

where

$$S_E(x(t)) \stackrel{\text{def}}{=} \int_{t_a}^{t_b} \left[\frac{m}{2}x'(t)^2 + V(x(t))\right] dt \tag{13.1.2}$$

is the *Euclidean action integral* of the path $x(t)$; this corresponds to the action integral of our particle of mass m moving under the influence of the *inverted potential* $-V$. By (13.1.1)

$$\exp\left(\frac{iS(x(t))}{\hbar}\right) \to \exp\left(-\frac{S_E(x(t))}{\hbar}\right) \tag{13.1.3}$$

and thus, in comparison with (11.2.5), we denote the Euclidean path integral $I_E(t_a, t_b, x_a, x_b)$ (not yet defined) by

$$I_E(t_a, t_b, x_a, x_b) = \int_{x_a}^{x_b} \exp\left(-\frac{S_E(x(t))}{\hbar}\right) D[x(t)]. \tag{13.1.4}$$

We also declare that $t_b - t_a \to -i(t_b - t_a)$ so that in (12.1.27)

$$c_n \to \widetilde{c}_n \stackrel{\text{def}}{=} \left[\frac{mn}{2\pi\hbar(t_b - t_a)}\right]^{n/2}. \tag{13.1.5}$$

Given (13.1.2) we have the reasonable approximation (for large n)

$$S_E(x^{(n)}(t)) \simeq \sum_{j=0}^{n-1} \left[\frac{m}{2\epsilon_n}(x_{j+1} - x_j)^2 + \epsilon_n V(x_{j+1})\right], \tag{13.1.6}$$

as in (11.2.11). It is now clear by comparison with (11.2.12), (11.2.13), how to define $I_E(t_a, t_b, x_a, x_b)$ in (13.1.4). Namely

$$I_E(t_a, t_b, x_a, x_b) \stackrel{\text{def}}{=} \lim_{n\to\infty} I_{E,n}(t_a, t_b, x_a, x_b), \tag{13.1.7}$$

where for $\widetilde{c}_n$ in (13.1.5)

$$I_{E,n}(t_a, t_b, x_a, x_b) \stackrel{\text{def}}{=} \widetilde{c}_n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-\frac{S_E(x(t))}{\hbar}\right) dx_1 dx_2 \cdots dx_{n-1} \tag{13.1.8}$$

with $S_E(x(t))$ regarded as a function of $x_1, x_2, \ldots, x_{n-1}$, given by the right-hand side of (13.1.6).

As a simple example consider again the free particle of mass m: $V = 0$. In this case we have

Proposition 13.1.

$$I_E(t_a, t_b, x_a, x_b) = \sqrt{\frac{m}{2\pi\hbar(t_b - t_a)}} \exp\left(-\frac{m}{2\hbar}\frac{(x_a - x_b)^2}{(t_b - t_a)}\right).$$

This compares with Theorem 11.4.

Proof. The Proposition follows directly from Theorem 11.6, just as Theorem 11.4 followed from Theorem 11.3. □

As a second example, which is non-trivial, consider again the simple harmonic oscillator with frequency ν and potential $V(x) = m\omega^2 x^2/2$ where $\omega = 2\pi\nu$. To compute $I_E(t_a, t_b, x_a, x_b)$ we follow exactly the methods of Chapter 12, setting up the appropriate analogous results. First of all we need a classical path $x_c(t)$ analogous to the path $x(t) = A\sin(\omega t + \phi)$ in (12.1.1) there for V. Since the Euclidean action functional S_E corresponds to the inverted potential $-V$ (as noted