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Critical dynamics in the early universe

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Abstract. Methods and concepts for the study of phase transitions mediated by a time-dependent order-parameter field in curved spacetimes are discussed. A practical example is the derivation of an effective (quasi-)potential for the description of 'slow-roll' inflation in the early universe. We first summarize our early results on viewing the symmetry behavior of constant background fields in curved but static spacetimes as finite size effect, and the use of derivative expansions for constructing effective actions for slowly-varying background fields. We then introduce the notion of dynamical finite size effect to explain how an exponential expansion of the scale factor imparts a finite size to the system and how the symmetry behavior in de Sitter space can be understood qualitatively in this light. We reason why the exponential inflation can be described equivalently by a scale transformation, thus rendering this special class of dynamics as effectively static. Finally we show how, in this view, one can treat the class of 'slow-roll' inflation as a dynamic perturbation off the effectively static class of exponential inflation and understand it as a dynamical critical phenomenon in cosmology.

1. Effective action for dynamic order-parameter fields

In this talk I would like to report on some new thoughts on the question of how to construct an effective action for a slowly-varying order-parameter field for the description of a class of inflationary cosmologies [1] where the transition to the true vacuum takes place via a gradually changing potential, such as the 'slow-roll' type in new inflation [2]. Knowledge of the exact form of the effective action holds the key to a complete description of a phase transition. One can deduce not only the qualitative features (first or second order) but also the quantitative details (mechanisms and processes). There has been recent interest in understanding the nature and construction of the effective potentials for scalar fields in different cosmological spacetimes. The interest is both theoretical, when viewed as a problem of finding the infrared behavior of quantum fields in curved spacetimes, and practical, when it is applied to descriptions of phase transitions in the very early universe (from the Planck time to the GUT time), or even the late universe (e.g., for electroweak phase transitions).

The work I am reporting to you now is a continuation of the program of study I began almost ten years ago with O'Connor and Shen on the symmetry behavior of quantum fields in curved spacetime. The first stage of our work centered on the simpler category of static spacetimes and constant order-parameter fields. We found that both geometrical and topological factors influence the infrared behaviour (IR) and often can be thought of as finite size effects. For a review of this first stage of our work, see [3, 4, 5]. The implication of finite size effect on cosmological phase transitions have been discussed for the de Sitter and mixmaster universes in [6, 7].

Most theoretical studies of inflation are based on the effective potential of the inflaton field in the de Sitter universe. It is not difficult to carry out such a calculation, as the mode functions are known explicitly [8]. However, one needs to be careful in correctly incorporating the influence of the zero mode and the higher modes of the spectrum on the critical behavior. We used the 2-particle irreducible (2PI) formalism [9], which involves the consistent solution of an equation for the background field and one for the two-point function[4]. This helped to decipher the infrared behavior near the critical point (for a massless, minimally coupled field in a symmetric vacuum). We actually carried out such a program for a general class of product spaces, including cosmological spacetimes as well as the Kaluza-Klein and the imaginary-time finite temperature field theories. These techniques and results should be useful for tackling a wide range of problems not necessarily related to curved spacetime. A problem of current interest is to work out the fine structure of the finite temperature effective potential for Higgs fields near the symmetric vacuum in the electroweak phase transition in connection with, e.g., late-time baryogenesis processes.

Although we can get a reasonable intuitive description of the physics in the above situations, quantitative statements are still unreliable because of the persistence of IR problems (even if the 2PI effective action is used). The essential reason for this is that the microscopic renormalized parameters are no longer adequate for a description of the IR physics. For example, in the finite temperature case a perturbative analysis in terms of the $4 - d$ parameters is well known to be plagued with IR problems. These IR problems lead to a breakdown of perturbation theory in terms of the $4 - d$ parameters and are symptomatic of the fact that one is trying to describe essentially $3 - d$ physics in terms of $4 - d$ physics. In other words, the effective degrees of freedom in the problem are changing as a function of scale. Since 1990 Denjoe O'Connor and Chris Stephens have developed a quantitative formalism wherein these IR problems are controlled. They have applied their techniques to a broad range of problems in different areas of physics with new findings on dimensional reduction and cross-over behaviors. The essence of their work is the development of a renormalization group (RG) that can interpolate between qualitatively different degrees of freedom[10, 11]. For the finite temperature case their RG is explicitly temperature (T) dependent, and acts in such a way that for T near zero it effectively integrates out $4 - d$ degrees of freedom. Such an RG follows as closely as possible the action of the dilation generator of true scale changes. (See their contributions in this volume for a recent summary.)

The second stage of our work on phase transitions in the early universe with time-varying background fields began in 1987 with Sukanya Sinha and Yuhong Zhang. We concentrated on situations where the order parameter field changes either with space or time. A familiar example in condensed matter physics is anisotropic superconductivity where one can use a gradient expansion in the Landau-Ginzberg-Wilson effective potential to account for the differences coming from the next-to-nearest neighbor interactions. For cosmological problems, it is the time-dependence of the background field which one needs to deal with. Strictly speaking, phase transition studies usually carried out assuming a constant field in the de Sitter universe is unrealistic, in that it only addresses the situation after the universe has entered the inflationary stage and inflates indefinitely. This model cannot be used to answer questions raised concerning the likelihood that the universe will still inflate if it had started from a more general, less symmetric initial state, such as the mixmaster

universe [12, 13, 14]. Nor can one use this model to study the actual process of phase transition (e.g., slow roll-over), and the problem of exit (graceful or not) to the 'true' Friedmann phase. To do this, as is well-known, one usually assumes that the potential is not exactly flat, but has a downward slope which enables the inflaton field to gradually (so as to give sufficient inflation) settle into a global ground state. The cosmological solution is, of course, no longer a de Sitter universe. Thus for a realistic description of many inflationary transitions one needs to treat the case of a nonflat potential and a time-varying field. The form of the potential and the metric of the background spacetime determine the behavior of the scalar field in the Laplace-Beltrami equation, but the field in turn provides the source of the Einstein equation which determines the behavior of the background spacetime metric. Hence they ought to be solved self-consistently. (One usually considers only the homogeneous mode of the scalar field for the dynamics of inflation and the inhomogeneous modes of quantum fluctuations for structure formation.) At the classical level, the wave equation for the background scalar field (assumed homogeneous) with self-interaction potential $V(\phi)$ in a spatially-flat Robertson-Walker (RW) spacetime is given by

$$\ddot{\phi} + 3H(t)\dot{\phi} + V'(\phi) = 0 \quad \dot{H} + 3H^2 = 8\pi G V(\phi) \quad \dot{H} = -4\pi G \dot{\phi}^2 \quad (1)$$

where $H(t) \equiv \dot{a}/a$ is the Hubble rate, a dot denotes derivative with respect to cosmic time t , and a prime denotes a derivative taken with respect to its argument. A trivial but important solution to these equations is obtained by assuming that $V(\phi) = V_0 = \text{constant}$, $\phi = \phi_0 = \text{constant}$ and $H = H_0 = \text{constant}$, which is the de Sitter universe $a = e^{H_0 t}$ with a constant field. A less trivial but useful solution is the so-called 'power-law' inflation models [15], with an exponential potential and a slowly-varying inflaton field (see equations (3, 4) below). One can take this as an example of a 'slow-roll' transition. The methods we have devised can be used to derive the effective potential (strictly speaking, quasi-potential) of such classes of spacetimes and fields. Let me describe in the following sections the theoretical difficulties and ways to overcome them. Records of the second stage of our research can be found in [16, 17, 18]. †

† Before closing this introduction, let me make a comment on the meaning of the term 'critical dynamics' as used in the context of cosmological phase transitions. By it we refer to studies of phase transitions mediated by a time-dependent order parameter field in contrast to static critical phenomena where the order-parameter field is constant in time. We are using this term in a general sense, not necessarily referring to the specific conditions of critical phenomena as discussed in condensed matter systems [19]. For example, critical phenomena usually deals with the change of the order parameter field near the critical point as a function of temperature. In cosmology, temperature T is a parameter usually (e.g. under the assumption of adiabatic expansion) tied in with the scale factor a and does not play the same role as in critical phenomena. In the new inflationary scenario, the critical temperature T_c is defined as the temperature at which a global ground state (the true vacuum) first appears. The stage when vacuum energy begins to dominate and inflation starts is the commencement of phase transition. The stage when the system begins to enter the true vacuum and reheat can be regarded for practical purpose as the end of phase transition. Throughout the process of inflation the system is in a 'critical' state. The progression of a cosmological phase transition is measured not by temperature, but by the change of field configurations in time. Criticality corresponds to the physical condition that the correlation length $\xi = m_{eff}^{-1} \rightarrow \infty$, or $m_{eff}^2 = d^2V/d\phi^2|_{\phi=0} \rightarrow 0$ which may or may not be possible). In critical dynamics studies of condensed matter systems one usually analyses the time-dependent Landau-Ginzberg equation, with a noise term representing the effect of a thermal bath and studies how the system (order-parameter field) settles into equilibrium as it approaches the critical point. We are not concerned with the corresponding cosmological problem here. An attempt to describe this aspect of inflationary transition was made in [20]. See also [21].

2. Quasistatic approximation versus dynamical finite size effect

The effective potential $V(\phi)$ gives a well defined description of phase transition only for a constant background (order-parameter) field. If the order-parameter field is dynamic, the effective potential is ill-defined and a host of problems will arise. Indeed, the very meaning of phase transition can become questionable. This is because as the field changes the effective action functional changes, and the locations of the minima change also. The notion of symmetry breaking and restoration is meaningful only when there exist well-defined global and local minima which does not change much in the time scale of the phase transition. Changing background field will also engender particle creation, which affects the nature and energetics of phase transition as well. Therefore, in the context of phase transitions involving dynamic fields, short of creating a new framework, one can at best discuss the problem in a perturbative sense, where the background field is nearly constant (quasi-static), so that an effective quasi-potential can still be defined [22, 17]. An effective Lagrangian for a slowly-varying background field can be obtained by carrying out a quasilocal expansion in derivative orders of the field, the leading term being the effective potential [23].

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi, \dots) \quad (2)$$

One can use this method to derive effective quasi-potentials for scalar fields in flat space (For an example of its application to electroweak finite temperature transition see the recent paper by Moss et al [24]), or (in conformal time) for the conformally-flat Robertson-Walker spacetimes. This is useful for studying cosmological phase transitions where the background spacetime changes only gradually, as in the Friedmann (low-power law) solutions $a = t^p, p < 1$. (For a description see [17].) However, for the inflationary universe where the scale factor undergoes rapid expansion following either an exponential $a = e^{Ht}$ or a high power-law behavior, the quasilocal expansion which assumes that the background field varies slowly is usually inadequate. That was the quandary we were in until the idea of using scaling to describe inflation dawned upon us. Viewed in this new light the de Sitter exponential expansion can be regarded as effectively static.

The first lead to such a connection came from our earlier investigation into the infrared behavior of quantum fields in de Sitter universe. If one follows the main results we obtained for static spacetimes and view the de Sitter space in the S^4 coordinatization, one can easily come up with a fairly good qualitative depiction. In our earlier work we introduced the concept of an effective infrared dimension (EIRD) [4]. This concept has been generalized and given a quantitative meaning in the work of O'Connor and Stephens [25] where they define an effective dimension (d_{eff}) which for de Sitter space is a function of $\eta \equiv \xi/a$ (see footnote 1 for the definition of ξ) and varies between 4 and a number close to zero. The reason it cannot go to zero is because there is a maximum value for the correlation length in de Sitter space and the RG must stop running at this value. This in fact is a generic feature of the RG for totally finite systems. However, if one views de Sitter space in other coordinatizations, such as the $S^3 \times R^1$ or the $R^3 \times R^1$ cases, one would have quite a different description of the physics where the obvious connection with a finite geometry is lost. We know that physics should be the same despite differences in coordinate descriptions. The resolution of this puzzle brings in an interesting point on the effect of spacetime dynamics on the symmetry behavior of a quantum field.

Specifically, for the special case of exponential expansion, in the spectrum of the 4-dim (spacetime) wave operator, there is a gap which separates the zero mode from the rest. This is what gives the $d_{eff} \simeq 0$ for the deep IR behavior of the scalar field in these other coordinate descriptions. Physically this arises from the fact that at late times, as a result of exponential expansion, most of the high-lying modes are stacked-up against the zero mode by the rapid red-shifting. (The S^3 or R^3 spatial sector also becomes immaterial.) The appearance of a scale (the event horizon H^{-1}) is a unique feature of this exponential class of expansion. It gives rise to effects identical to that originating from some finite size in some associated static spacetimes. This is why we refer to these effects as 'dynamical finite size effects' [16, 5].

3. Inflation as scaling: static critical phenomena

The other lead came from the work I did with Yuhong Zhang in 1990 [18] on coarse-graining and backreaction in stochastic inflation. There, in trying to compare the inflationary universe with phase transitions in the Landau–Ginzberg model, using a $\lambda\phi^4$ field as example, we realize that the exponential expansion of the scale factor can be viewed as the system undergoing a Kadanoff–Migdal scale transformation [26]. This can be seen as follows: Consider a spatially-flat RW metric with a constant scale factor. This is just the Minkowski spacetime. Let us consider an ordered sequence of such static hyperspaces (foliation) with scales a_0, a_1, a_2 , etc parameterized by $t_n = t_0 + n\Delta t, n = 0, 1, 2, \dots$. These spacetimes have the same geometry and topology but differ only in the physical scale in space. One can always redefine the physical scale length $x'_{(n)} = a_n x$ to render them equivalent. If each copy has scale length magnified by a fixed factor s over the previous one in the sequence, i.e. $a_{n+1}/a_n \equiv s = e^{H\Delta t}$, we get exactly the physical picture as in an eternal inflation. Here s is independent of time. After n -iterations i.e. $a_n/a_0 = e^{n(H\Delta t)}$, or, with a continuous parameter $a(t) = a_0 e^{Ht}$. It is important to recognize that t can be any real parameter not necessarily related to time. In other words, time in this case plays the role of a scaling parameter. It does not have to be viewed as a dynamical parameter. Thus for this special class of expansion, the dynamics of spacetime can be replaced equivalently by a scaling transformation. In so doing one renders eternal inflation into a static setting. By contrast, the larger class of power-law expansion $a = t^\gamma$ does not possess this scaling property. We see that $a_n/a_0 = (1 + n\Delta t/t)^\gamma$ depends on time.' Hence they cannot be viewed as effectively static. A useful parameter which marks the difference between these two classes of dynamics is $\zeta = |\dot{H}|/H^2 = \ddot{\alpha}/\dot{\alpha}^2$, where $\alpha \equiv \ln a$, which can be regarded as a 'nonadiabaticity parameter' of dynamics: the de Sitter exponential behavior with $\zeta = 0$ is 'static', the slow-roll with small $\zeta \ll 1$ is 'adiabatic', while the RW low-power-law with $\zeta \approx 1$ is 'nonadiabatic'. This rather unusual characteratization is nevertheless quite useful. It captures the essence of our considerations above in distinguishing between static vs dynamical finite size effect and static vs dynamic scaling [19]. As distinct from the rather unique de Sitter case, where only one parameter, the scaling parameter s , is needed for the description of the dynamics of spacetime and the field, in the general class of RW dynamics, two parameters are required : the scaling parameter s which describes inflation, and a dynamic parameter t which describes the evolution of the field different from the 'static' (eternal inflation) case. These two parameters appear

also in the dynamical renormalization group theory description of dynamical critical phenomena.

4. ‘Slow-roll’ as dynamical critical phenomena

Using the conceptual framework introduced above, one can understand why the particular subclass of high-power-law expansion associated with an exponential potential can hence be viewed as quasi-static, as it differs only slightly from the exponential expansion in their qualitative behavior. It is in this context that one can once again introduce the quasilocal approximation to derive the effective action for scalar fields to depict this more realistic ‘slow-roll’ inflation, now carried out as a quasilocal perturbation from the de Sitter space, which is viewed as effectively static.

A classical solution to (2) is given by [15]:

$$V(\phi) = V_0 \exp\left(-\frac{\epsilon}{l_p} \phi\right) \quad a(t) = a_0(1 + H_0 \Delta t / \gamma)^\gamma \quad (3)$$

Here $l_p = 1/\sqrt{8\pi G}$ is the Planck length, $V_0 = [(3\gamma - 1)/\gamma](H_0^2/8\pi G)$, $\gamma = 2/\epsilon^2$ and $\Delta t = t - t_0$. The subscript 0 denotes the value at an initial time t_0 where the de Sitter solution holds. Note also that for $\gamma \rightarrow \infty$ or $\epsilon \rightarrow 0$, we get the class of de Sitter solution $a(t) = a_0 e^{H_0(t-t_0)}$. In that limit, the scaling parameter $s = [a(t + t_0)/a(t)] = (1 + \sigma/\gamma)^\gamma$ goes over to e^σ , where $\sigma = H\Delta t$.

The time-dependence of the scalar field ϕ associated with the above solutions of V and a is given by [15]

$$\phi(t) = \phi_0 + \sqrt{\gamma/4\pi G} \ln(1 + H\Delta t/\gamma) \quad (4)$$

Now, following the rationale we suggested above, we can view this solution as a ‘quasi-static’ generalization of the de Sitter solution. Expanding $\phi(t)$ for small time difference from t_0 , we get,

$$\phi(t) - \phi_0 = (\epsilon l_p H_0) \Delta t - \left(\frac{\epsilon^3}{8} l_p^2 H_0^2\right) (\Delta t)^2 + \dots \quad (5)$$

Here we can identify the coefficients of the first two terms $(\Delta t)^n$, $n = 1, 2$ as the leading coefficients in a quasi-local expansion of the effective mass which involves the background field [23]. After such an identification, it is easy to adopt the well-established derivative expansion scheme for the calculation of the quantum effective quasi-potential. The general result is given in [17]. There, as we recall, the effective Lagrangian contains the ‘kinetic energy’ terms as well as the radiative corrections arising from the varying background field. Details of the derivation and discussions on the physical meaning of these results in ‘slow roll-over’ inflations are given in [27]. (Kay Pirk has recently derived a quantum solution to (3) [28].)

5. Summary

Let me summarize the main points brought up in this talk by the following schematic diagram:

Constant field in static or conformally static spacetimes (Finite size effect)	SCALING →	Exponential expansion $a = e^{Ht}$ 'Eternal inflation' (Dynamical finite size effect)
<i>Quasilocal approximation</i>		<i>Derivative expansion</i>
Slowly varying field in RW universe (Low power-law)	→	High power-law expansion $a = t^p$ 'Slow roll-over'

For slowly-varying background fields one can use the method of derivative expansion to derive the quasilocal effective Lagrangian. Usually this makes sense only for static (or conformally-static, like the RW) spacetimes. However, one can view the special class of exponential expansion as effectively 'static'. This can be understood with the ideas of 'dynamical finite size effect' and implemented by treating inflation as 'scaling' transformations. The 'slow roll-over' type of phase transition used in many inflationary models can be viewed as a quasi-static case, and derived as a dynamic perturbation from the de Sitter universe. An example which this reasoning can be applied to for the calculation of the effective 'quasi-potential' is the case of a high-power-law expansion with an exponential potential for the inflaton field.

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References

- [1] Guth A 1981 *Phys Rev. D* **23** 347
- [2] Linde A 1982 *Phys Lett.* **108B** 389
Albrecht A and Steinhardt P J 1982 *Phys Rev. Lett.* **48** 1220
- [3] Hu B L 1986 *Proc. 4th Marcel Grossmann Meeting on General Relativity (Rome, 1985)* ed R Ruffini (Amsterdam: North-Holland)
- [4] Hu B L and O'Connor D J 1987 *Phys Rev. D* **36** 1701
- [5] O'Connor D J, Stephens C R and Hu B L 1990 *Ann. Phys., NY* **190** 310
- [6] Hu B L and O'Connor D J 1986 *Phys Rev. Lett.* **56** 1613
- [7] Hu B L and O'Connor D J 1986 *Phys Rev. D* **34** 2535
- [8] Shore G 1980 *Ann. Phys., NY* **128** 376
Hawking S W and Moss I 1982 *Phys Lett.* **110B** 35
Vilenkin A 1983 *Nucl. Phys. B* **226** 504

- Allen B 1983 *Nucl Phys.* B 226 228
 Ratra B 1985 *Phys Rev.* D 31 1931
- [9] Cornwall J, Jackiw R and Tomboulis D 1974 *Phys Rev.* D 10 2428
 [10] O'Connor D and Stephens C R 1991 *Nucl Phys.* B 260 3652
 [11] O'Connor D and Stephens C R 1993 *Geometry the Renormalization Group and Gravity Misner Festschrift* ed B L Hu, M P Ryan and C V Vishveshwara (Cambridge: Cambridge University Press) to appear
 [12] Wald R M 1983 *Phys Rev.* D 28 2118
 [13] Boucher W, Gibbons G W and Horowitz G T 1984 *Phys Rev.* D 30 2447
 [14] Shen T C, Hu B L and O'Connor D J 1985 *Phys Rev.* D 31 2401
 [15] Lucchin F and Matarrese S 1985 *Phys Rev.* D 32 1316
 [16] Hu B L 1988 *Proc. CAP-NSEERC Summer Inst. in Theoretical Physics* vol II, ed F C Khanna, H Umezawa, G Kunstatter and P Lee (Singapore: World Scientific)
 [17] Sinha S and Hu B L 1989 *Phys Rev.* D 38 2423
 [18] Hu B L and Zhang Y 1990 Coarse-Graining, Scaling, and Inflation *University of Maryland Preprint* 90-186
 Hu B L 1991 *Relativity and Gravitation: Classical and Quantum: Proc. SILARG VII (Cocoyoc, Mexico 1990)* ed J C D' Oliva et al (Singapore: World Scientific)
 [19] See, e.g., Hohenberg P C and Halperin B I 1977 *Rev Mod. Phys.* 49 435
 Ma S K 1976 *Modern Critical Phenomena* (New York: Benjamin)
 Kawasaki K and Guntun J 1976 *Phys Rev.* B 13 4658
- [20] Cornwall J M and Bruinsma R 1988 *Phys Rev.* D 38 3146
 [21] Hu B L, Paz J P and Raval A in preparation
 [22] Hu B L 1983 *Phys Lett.* 123B 189
 Hu B L and Chen L F 1985 *Phys Lett.* 160B 36
 [23] Hu B L, and O'Connor D J 1984 *Phys Rev.* D 30 743
 [24] Moss I, Toms D J and Wright A 1992 *Phys Rev.* D 46 1670
 [25] O'Connor D and Stephens C R in preparation
 [26] Kadanoff L 1966 *Ann Phys., NY* 2 263
 Migdal A A 1975 *Sov Phys.-JETP* 69 810; L457
 [27] Hu B L and Sinha S in preparation
 [28] Pirk K 1992 *University of Maryland preprint*