A Depth on the Maggiore's Commutation Relations

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The generalized commutation relations (GCRs) extend the familiar Heisenberg commutation relations and have garnered significant attention due to their broad applicability. Various generalizations have been proposed and are currently the subject of extensive investigation. It is crucial to scrutinize the physical implications of these GCRs, which reveal intriguing and complex phenomena. While the GCRs describe the relationships between observables, they solely pertain to kinematics. The dynamics, on the other hand, is dictated by the Hamiltonian operator H, which lacks a unique determination from fundamental physical principles. By analyzing two distinct choices for H, it was discovered that these GCRs naturally lead to phenomena such as variable speed of light, modified dispersion relations, and a reduction in thermodynamic degrees of freedom. These findings underscore the inherent interest and significance of Maggiore's GCRs.

The conventional Heisenberg uncertainty principle, which relates the uncertainties in the position and momentum of a particle, can be extended in various ways in multidimensional space. Maggiore introduced a generalized set of commutation relations (GCRs) between position operators X_i and momentum operators P_j for three-dimensional space under specific assumptions. These assumptions preserve the undeformed nature of spatial rotations and translations while allowing for deformation in the position and momentum commutators controlled by a parameter λ. Extending this to arbitrary dimensions, Maggiore's GCRs are given as:

$$[X_i, X_j] = -i\hbar f \tilde{J}_{ij}, \quad [X_i, P_j] = i\hbar f \delta_{ij} \tag{1}$$

Here, $\hbar = 2\pi$ is the reduced Planck constant, $c_0 = 3 \times 10^8$ m/s is the speed of light in vacuum, and m is the particle mass. The function f and its modified counterpart $\tilde{f}$ are determined by the deformation parameter λ and the particle's momentum P and mass m as follows:

$$f = \frac{1+\lambda^2}{\hbar^2}\left(1 + \frac{\lambda^2 P^2}{\hbar^2 + m^2 c_0^2}\right), \quad \tilde{f} = \frac{\lambda^2}{4\pi^2} \tag{2}$$

These relations imply non-commutativity of space at scales of $O(\lambda)$ or smaller. Kempf proposed another generalization where the position commutator vanishes, resulting in a commutative space. The energy and momentum scales set by λ, denoted by E^* and p^*, respectively, are given by:

$$E^* = p^* c_0 = \frac{hc_0}{\lambda}\sqrt{mc_0^2} \tag{3}$$

In the low energy or temperature limit, $E \ll E^*$, and in the high energy or temperature limit, $E \gg E^*$, the behavior of the system is determined by λ. The kinematics are governed by the commutation relations, while the dynamics is described by the Hamiltonian H. The velocity operator V_i for a particle is given by:

$$V_i = \frac{dX_i}{dt} = \frac{i}{\hbar}[H, X_i] \tag{4}$$

For free particles, H depends only on P^2 or equivalently P^2+m^2. The speed v of free particles is determined by the eigenvalues of V_i and P and is related to the energy E via:

$$v(E) = \frac{\sqrt{\sum_{i=1}^{d} v_i^2}}{p} = \frac{p}{E}\sqrt{p^2+m^2} \tag{5}$$

The statistical mechanics of such particles in a d-dimensional volume V can be studied using the grand canonical ensemble approach. Thermodynamic quantities like free energy (F), pressure (P), and internal energy (U) can be calculated using standard expressions. The one-particle density of states $g(E)$ is given by:

$$g(E) = \frac{\Omega_{d-1} V}{h}\frac{1}{v(E)} \tag{6}$$

Here, Ω_n represents the area of a unit n-dimensional sphere.

To proceed with the analysis, one needs to express the variables p and f, in terms of the energy E. This necessitates expressing them using the Hamiltonian $H(p)$ or, equivalently, the energy $E(p)$. However, as far as the current understanding goes, there isn't a physical principle that uniquely determines the energy $E(p)$ in the given context. In the absence of such a principle, one can impose two conditions: first, that the energy $E(p)$ approaches a finite value as λ approaches to zero, and second, that $E(p)$ approaches infinity as p approaches infinity. These conditions are reasonable from a physical standpoint but do not uniquely determine $E(p)$. Nevertheless, it's found that by examining a few representative choices of $E(p)$ which fulfill these conditions, one can illustrate the general dynamical behaviors of the system and their dependence on $E(p)$. The implications of the generalized cosmic rays (GCRs) are negligible in the low energy, low-temperature regime since $E(p)$ is finite-valued. Therefore, the focus is solely on the high energy, high-temperature limit, where the dynamics governed by the GCRs become evident. Following, the analysis will be on these dynamics for three specific choices of $E(p)$. The dominant terms in various quantities, up to numerical factors, suffice to demonstrate the system's dynamical features. For comparison, one also includes results for the standard case (designated as 0) where $\lambda = 0$ and $f = 1$, specifically focusing on the high energy, high-temperature limit where $E \propto p$. These choices of $E(p)$ are primarily selected to elucidate the general dynamical features of the GCRs and their dependence on $E(p)$. The choices of $E(p)$ demonstrate various ways in which can be satisfied. Some of these choices may have natural origins. For instance, choice 1 with $n = 1$ could stem from the first Casimir operator, while choice 2 with $n = 1$ might result from assuming the validity of the standard Hamiltonian even when $\lambda \neq 0$. Choice 2 with an integer $n > 1$ may be related to higher derivative terms in an effective action. The choices presented mainly serve to highlight the general dynamical features of the GCRs and their dependence on $E(p)$. Examining the general features of $c_\lambda(E)$ one can deduce the following observations. The speed of light $c_\lambda(E)$ increases with the asymptotic growth of $E(p)$, leading to a decrease in $g^*(E)$. In units where $c_0 = 1$, $c_\lambda(E)$ approaches 1 for choice 1 with $n < 1$, equals 1 for choice 1 with $n = 1$, and diverges for choice 1 with $n > 1$, as well as for choices 2 and 3. The physical implications of $c_\lambda(E) \neq 1$ and its dependence on energy are discussed elsewhere. Moving on to the thermodynamic quantities, their behavior in the low-temperature limit remains unaffected since $E(p)$ is finite-valued. Therefore, one can focus on the high-temperature limit, setting $m = 0$ and $\mu = 0$ and considering cases where $a = -1$ and $a = 0$, corresponding to particles obeying Bose–Einstein and Maxwell–Boltzmann statistics, respectively. Results for the $a = 1$ case are formally similar to those for $a = 0$ in this limit. Explicit evaluation of the partition function $\ln Z$ is challenging, if not impossible. However, in the limit $\beta \ll \lambda$, the leading-order behavior of $\ln Z$ can be obtained relatively easily. One can derive the temperature dependence of relevant quantities, such as $-\beta F$ and βU. The method used allows to extract the leading-order contributions to these quantities, for the $a = -1$ and $a = 0$ cases, respectively. The temperature dependence of $-\beta F$ provides insights into the effective thermodynamic degrees of freedom in the system. Several general features are shown as follows. The first is that the reduction in the degrees of freedom increases with the faster asymptotic growth of $E(p)$. Furthermore, the degrees of freedom in choice 1 are formally equivalent to those in the standard case but with an effective dimension $d_{eff} = 1/n$. Then, compared to the standard case, the degrees of freedom are reduced in choices 2 and 3, and in choice 1 with $n > 1/d$, while they are increased in choice 1 with $n < 1/d$. It occurs a reduction in degrees of freedom in choice 1 with $n = 1$ resembles that found in string theory at temperatures much higher than the Hagedorn temperature (if $\lambda \sim$) string length. Also, the reduction in degrees of freedom in the $a = 0$ case for choice 2 with any n resembles that found in lattice theories with a finite number of Bose oscillators at each site, or in certain topological field theories with general coordinate invariance restored at short distances. Finally, considering $w = P/\rho$, where $\rho = U/V$ is the energy density, it's observed that for perfect fluids in the standard case, w is constant and must be less than 1 since the speed of sound $v_s < 1$ in units where $c_0 = 1$. However, in the present case, w can exceed 1, allowing for v_s to also exceed 1 but always remaining less than c_λ.

In the early stages of the universe, radiation dominated, characterized by a scale factor $a = -1$ and following the Generalized Uncertainty Principle (GCRs). This era's evolution is described by standard equations considering radiation pressure P and energy density $\rho = U/V$, where U is the internal energy and V is volume. The line element in this scenario is given by:

$$ds^2 = -c^2 \lambda dt^2 + A^2(t) \sum_{i=1}^{d} dX_i dX_i \tag{7}$$

Where c is the speed of light, λ is a parameter, t is time, $A(t)$ is the scale factor, and d is the number of spatial dimensions. The comoving horizon radius rh at a time $t_0 > 0$ is defined as:

$$rh = \int_{t_0}^{t \to 0} dt \frac{c\lambda}{A} \tag{8}$$

Here, $t = 0$ represents the big bang singularity time. By considering temperature T as the independent variable, the values of t, A, and rh can be calculated to the leading order. K is given by:

$$K \equiv 1 - \frac{2}{d(n+1)} - \frac{2(n-1)}{n} \tag{9}$$

Following, several general features are shown. As T approaches infinity, and A approaches zero as t approaches zero, indicating a big bang singularity. However, for specific choices of parameters, $A(t)$ decreases slower compared to the standard case. As T tends to infinity, for certain parameter choices, rh tends to infinity as well. This occurs for choice 1 if $K < 0$, which is equivalent to $n > n^*$, where $n^* = d - 2 + \sqrt{(d-2)^2 + 8d^2}/(2d)$. It's notable that $1 \leq n^* \leq 2$ for $1 \leq d \leq \infty$. Thus, rh tends to infinity for choice 1 if $n \geq 2$. Furthermore, if $c\lambda(E)$ increases at least as fast as $\sqrt{\lambda E}$ in the limit $\lambda E \to \infty$, then rh remains finite for choice 1 with $1 < n < n^*$, despite the increase in $c\lambda(E)$ with energy. This implies that the divergence of rh is not solely determined by the increase in the speed of light with energy; the scale factor A must also approach zero at a sufficiently slow rate. Thus, photons with a speed $c\lambda(E)$ close to unity have enough time to establish causal contact within the horizon before encountering the singularity. By studying the dynamical features of the GCRs and their dependence on energy $E(p)$ through different choices, one can do several observation. The dynamical quantities are largely independent of the number of spatial dimensions. Then, the black body spectrum converges to a limiting form independent of the choice of $E(p)$. While the speed of light varies with energy, with a larger speed for faster growth of $E(p)$, resulting in a smaller one-particle density of states. Further, thermodynamic relations undergo significant modifications, with the effective degrees of freedom depending on energy, and their reduction being faster for higher growth rates of $E(p)$. Also, in the early universe, the scale factor evolves slower, and the horizon size increases faster with a higher growth rate of $E(p)$, leading to a tendency for the horizon to expand indefinitely. Given these findings, further investigations into GCRs and their implications, particularly regarding modifications to standard Lorentz invariance and their potential impact on cosmology and black hole physics, are deemed promising. This could lead to a more comprehensive understanding and potentially a coordinate-invariant formulation of GCRs, facilitating rigorous studies of their cosmological and astrophysical consequences.